Integrals-tasks (II part)

Integration by substitution

If we introduce subtitution $x = g(t)$ then $dx = g`(t)dt$ and the initial integral $\int f(x)dx$ becomes:

$$\boxed{\int f(x)dx = \int f(g(t)) \cdot g`(t)dt}$$

For starters here's an advice: **for substitut choose the term which derivative is with dx.**

Simply stated, substitution means that in the given integral something we choose to be *t* (for example, Ω). nadjemo From that we find derivate and replace it in the initial integral , which is now all “by t” $\begin{vmatrix} \Omega = t \\ \Omega`dx = dt \end{vmatrix}$.

Examples:

1

$$\int \frac{2xdx}{x^2+12} = ?$$

We can see that we have the expression *2x* in front of *dx.* Thinking of which the derivative is *2x* ? We know that

$(x^2)`= 2x$ and we choose that as *t*. It is still smarter to take the entire expression x^2+12 for *t* (**substitut),** because:

$[(x^2+12)`= 2x]$

$$\int \frac{2xdx}{x^2+12} = \begin{vmatrix} x^2+12 = t \\ 2xdx = dt \end{vmatrix} = \int \frac{dt}{t} = \ln|t| + C = \text{when solving integral 'by t' },$$

then substitut back and get a solution 'by x' = $\boxed{\ln|x^2+12| + C}$

2

$$\int \frac{x^2dx}{x^3+1} = ?$$

Think like in the previous example : derivate from x^3+1 is $3x^2$, and

we know that the constant can always go in front of the integral becaose of rule $\boxed{\int A \cdot f(x)dx = A \cdot \int f(x)dx}$

which we explained in the previous part.

$$\int \frac{x^2dx}{x^3+1} = \begin{vmatrix} x^3+1 = t \\ 3x^2dx = dt \to x^2dx = \frac{dt}{3} \end{vmatrix} = \int \frac{\frac{dt}{3}}{t} = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}\ln|t| + C = \boxed{\frac{1}{3}\ln|x^3+1| + C}$$

3

$$\int \frac{1}{x+5}dx = ?$$

This integral looks like a tablet $\int \frac{1}{x}dx = \ln|x| + C$ but instead of *x* in the denominator, we have *x + 5*.

$$\int \frac{1}{x+5}dx = \begin{vmatrix} x+5=t \\ dx = dt \end{vmatrix} = \int \frac{1}{t}dt = \ln|t| + C = \boxed{\ln|x+5| + C}$$

Related to these integrals, we perform one conclusion: $\boxed{\int \frac{1}{x \pm a}dx = \ln|x \pm a| + C}$

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$$\int \frac{1}{(x+5)^3}dx = ?$$

$$\int \frac{1}{(x+5)^3}dx = \begin{vmatrix} x+5=t \\ dx = dt \end{vmatrix} = \int \frac{1}{t^3}dt = \int t^{-3}dt = \frac{t^{-3+1}}{-3+1} + C = -\frac{1}{2\cdot t^2} + C = -\frac{1}{2\cdot(x+5)^2} + C$$

5

$$\int \sin^2 x \cdot \cos x dx = ?$$

We know that (sinx)`=cosx , and :

$$\int \sin^2 x \cdot \cos x dx = \begin{vmatrix} \sin x = t \\ \cos x dx = dt \end{vmatrix} = \int t^2 dt = \frac{t^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}$$

6

$$\int e^{-x^3} x^2 dx = ?$$

$$\int e^{-x^3} x^2 dx = \begin{vmatrix} -x^3 = t \\ -3x^2 dx = dt \rightarrow x^2 dx = \frac{dt}{-3} \end{vmatrix} = \int e^t \frac{dt}{-3} = -\frac{1}{3}\int e^t dt = -\frac{1}{3}e^t + C = \boxed{-\frac{1}{3}e^{-x^3} + C}$$

7

$\int ctgx dx = ?$

You must first use the identity $ctgx = \frac{\cos x}{\sin x}$,

$$\int ctgx dx = \int \frac{\cos x}{\sin x} dx = \left| \begin{matrix} \sin x = t \\ \cos x dx = dt \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + C = \ln|\sin x| + C$$

8

$\int \frac{arctgy}{1+y^2} dy = ?$

$$\int \frac{arctgy}{1+y^2} dy = \left| \begin{matrix} arctgy = t \\ \frac{1}{1+y^2} dy = dt \end{matrix} \right| = \int t dt = \frac{t^2}{2} + C = \boxed{\frac{(arctgy)^2}{2} + C}$$

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$\int \frac{x^2 dx}{x^6 + 4} = ?$

$$\int \frac{x^2 dx}{x^6 + 4} = \int \frac{x^2 dx}{(x^3)^2 + 4} = \left| \begin{matrix} x^3 = t \\ 3x^2 dx = dt \to x^2 dx = \frac{dt}{3} \end{matrix} \right| = \int \frac{\frac{dt}{3}}{t^2 + 4} = \frac{1}{3} \int \frac{dt}{t^2 + 4}$$

Here we use a tablet integral $\int \frac{1}{a^2 + t^2} dx = \frac{1}{a} arctg \frac{t}{a} + C$ but we must first determine *a:*

$$= \frac{1}{3} \int \frac{dt}{t^2 + 4} = \frac{1}{3} \int \frac{dt}{t^2 + 2^2} = \frac{1}{3} \cdot \frac{1}{2} arctg \frac{t}{2} + C = \boxed{\frac{1}{6} arctg \frac{x^3}{2} + C}$$

When we have already used this tablet integral, if you remember, we mentioned that all teachers do not allow to be on used. Pa da vidimo kako smo mi njega rešili metodom smene: So let's see how we solve it using substitution.

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$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}arctg\frac{x}{a}+C$$

Proof:

$$\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2[1+(\frac{x}{a})^2]}dx = \frac{1}{a^2}\int \frac{1}{[1+(\frac{x}{a})^2]}dx = \left| \begin{array}{c} \frac{x}{a}=t \\ \frac{dx}{a}=dt \rightarrow dx = adt \end{array} \right| = \frac{1}{a^2}\int \frac{1}{[1+t^2]}adt = \frac{1}{a^{\not 2}}\not a\int \frac{1}{[1+t^2]}dt =$$

$$\frac{1}{a}\int \frac{1}{[1+t^2]}dt = \frac{1}{a}arctgt + C = \boxed{\frac{1}{a}arctg\frac{x}{a}+C}$$

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$$\int \frac{1}{25+x^2}dx = ?$$

$$\int \frac{1}{25+x^2}dx = \int \frac{1}{5^2+x^2}dx = [\text{ovde je dakle } a\text{=5}] = \boxed{\frac{1}{5}arctg\frac{x}{5}+C}$$

A similar situation is to:

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$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}+C$$

Proof:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{a^2[1-(\frac{x}{a})^2]}}dx = \int \frac{1}{\sqrt{a^2}\cdot\sqrt{[1-(\frac{x}{a})^2]}}dx = \int \frac{1}{a\cdot\sqrt{[1-(\frac{x}{a})^2]}}dx = \frac{1}{a}\int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}dx$$

$$= \left| \begin{array}{c} \frac{x}{a}=t \\ \frac{dx}{a}=dt \rightarrow dx = adt \end{array} \right| = \frac{1}{a}\int \frac{1}{\sqrt{1-t^2}}adt = \frac{1}{\not a}\cdot\not a\cdot\int \frac{1}{\sqrt{1-t^2}}dt = \int \frac{1}{\sqrt{1-t^2}}dt = \arcsin t + C = \boxed{\arcsin\frac{x}{a}+C}$$

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$$\int \frac{1}{\sqrt{15-x^2}}\,dx = ?$$

$$\int \frac{1}{\sqrt{15-x^2}}\,dx = \int \frac{1}{\sqrt{(\sqrt{15})^2-x^2}}\,dx = [so:\ a=\sqrt{15}\ \text{ and then}] = \boxed{\arcsin\frac{x}{\sqrt{15}}+C}$$

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$\int \sin ax dx = ?$ where a is constant

$$\int \sin ax dx = \left| \begin{array}{l} ax = t \\ adx = dt \rightarrow dx = \dfrac{dt}{a} \end{array} \right| = \int \sin t \cdot \frac{dt}{a} = \frac{1}{a}\int \sin t dt = \frac{1}{a}(-\cos t)+C = \boxed{-\frac{1}{a}\cos ax + C}$$

A similar situation is to:

$$\int \cos ax dx = \frac{1}{a}\sin ax + C$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + C \qquad \text{etc.}$$

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$$\int \sin^2 x dx = ?$$

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}[\int 1\cdot dx - \int \cos 2x dx] = \frac{1}{2}[x-\frac{1}{2}\sin 2x]+C = \boxed{\frac{1}{2}x-\frac{1}{4}\sin 2x + C}$$

Remember the following examples: $\int \cos 2x dx = \frac{1}{2}\sin 2x$.

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$\int \cos^2 x dx = ?$

As is $\cos^2 x = \dfrac{1+\cos 2x}{2}$

$$\int \cos^2 x dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int [1+\cos 2x] dx =$$
$$= \frac{1}{2}[\int 1 \cdot dx + \int \cos 2x dx] = \frac{1}{2}[x + \frac{1}{2}\sin 2x] + C = \boxed{\frac{1}{2}x + \frac{1}{4}\sin 2x + C}$$

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$\int \dfrac{dx}{\sin x} = ?$

This task can be solved in multiple ways. We will use "trick":

$$\int \frac{dx}{\sin x} = \int \frac{dx}{\sin x} \cdot \frac{\sin x}{\sin x} = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1-\cos^2 x} dx$$

Now, we have:

$$\int \frac{\sin x}{1-\cos^2 x} dx = \begin{vmatrix} \cos x = t \\ -\sin x dx = dt \\ \sin x dx = -dt \end{vmatrix} = \int \frac{-dt}{1-t^2} = \int \frac{dt}{t^2-1}$$

This is a tablet integral $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$ so:

$$\int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C$$

The solution may remain as such, but we'll repair it for a purpose, this integral can be in elegant way solved through trigonometric subtitution, and there will be solution:

$$\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C = \ln\left|\frac{\cos x-1}{\cos x+1}\right|^{\frac{1}{2}} + C = \ln\left|\sqrt{\frac{\cos x-1}{\cos x+1}}\right| + C = \boxed{\ln\left|tg\frac{x}{2}\right| + C}$$

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$$\int \sqrt{\frac{1-x}{1+x}}dx = ?$$

$$\int \sqrt{\frac{1-x}{1+x}}dx = \int \frac{\sqrt{1-x}}{\sqrt{1+x}}dx = \text{Here is a trick made rationalization} = \int \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}}dx = \int \frac{1-x}{\sqrt{1-x^2}}$$

Now we separate the integral into two ..

$$\int \frac{1-x}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1-x^2}}dx - \int \frac{x}{\sqrt{1-x^2}}dx =$$

$$\int \frac{x}{\sqrt{1-x^2}}dx = \left| \begin{array}{l} \sqrt{1-x^2} = t \\ \frac{1}{\not{2}\sqrt{1-x^2}} \cdot (-\not{2}x)dx = dt \\ \frac{x}{\sqrt{1-x^2}} = -dt \end{array} \right| = \int (-dt) = -t + C = -\sqrt{1-x^2} + C$$

Return to the previous solution, and we have finale solution:

$$\int \frac{1-x}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1-x^2}}dx - \int \frac{x}{\sqrt{1-x^2}}dx = \arcsin x - (-\sqrt{1-x^2}) + C = \boxed{\arcsin x + \sqrt{1-x^2} + C}$$